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18. Sol:

$$\mathcal{L}[y^{(4)}-y] = s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = 0$$

$$= (s^4-1)Y(s) - (s^3+s) = 0$$

$$\Rightarrow Y(s) = \frac{s(s^2+1)}{(s^2+1)(s^2-1)} = \frac{s}{s^2-1} = \frac{1}{2}\left(\frac{1}{s-1}+\frac{1}{s+1}\right)$$

$$\Rightarrow y = \varphi(t) = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}$$

20. Sol:

$$\mathcal{L}[y''+\omega^2 y] = s^2 Y(s) - s y(0) - y'(0) + \omega^2 Y(s) = \mathcal{L}[\cos 2t]$$

$$\Rightarrow (s^2+\omega^2)Y(s) - s = \frac{s}{s^2+4}$$

$$\Rightarrow Y(s) = \cancel{\frac{s^2}{(s^2+\omega^2)(s^2+4)}}$$

$$= \frac{s}{(s^2+\omega^2)(s^2+4)} + \frac{s}{s^2+\omega^2}$$

$$= s\left\{\frac{1}{\omega^2-4}\left(\frac{1}{s^2+4}-\frac{1}{s^2+\omega^2}\right)+\frac{1}{s^2+\omega^2}\right\}$$

$$\Rightarrow y = \varphi(t) = \frac{1}{\omega^2-4}(\cos 2t - \cos\omega t) + \cos\omega t$$

$$= \frac{1}{\omega^2-4}\left((\omega^2-5)\cos\omega t + \cos 2t\right)$$

22. Sol:

$$\mathcal{L}[y''-2y'+2y] = \mathcal{L}[e^{-t}]$$

$$\Rightarrow s^2 Y(s) - s y(0) - y'(0) - 2s Y(s) + 2y(0) + 2Y(s) = \frac{1}{s+1}$$

$$\Rightarrow (s^2-2s+2)Y(s) = \frac{s+2}{s+1}$$

$$\Rightarrow Y(s) = \frac{1}{(s-1)^2+1} + \frac{-\frac{1}{5}s+\frac{3}{5}}{(s-1)^2+1} + \frac{\frac{1}{5}}{s+1} = \frac{8-s}{5[(s-1)^2+1]} + \frac{1}{5(s+1)}$$

i.e. $y = \varphi(t) = \frac{1}{5}\left(e^{-t} - e^{t}\cos t + 7 e^{t}\sin t \right)$

29. Sol:

a> $F'(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\, dt$

$= \int_0^\infty \frac{d}{ds} e^{-st} f(t)\, dt$ (by LDCT)

$= \int_0^\infty -t e^{-st} f(t)\, dt$

$= \mathcal{L}[-t f(t)]$

b> $F^{(2)}(s) = \mathcal{L}[(-t)\cdot(-t) f(t)] = \mathcal{L}[(-t)^2 f(t)]$

by induction, we may show that

$F^{(n)}(s) = \mathcal{L}[(-t)^n f(t)]$.

36. Sol:

a> $\mathcal{L}[ty] = -\frac{d}{ds}\mathcal{L}[y] = -Y'(s)$

$\mathcal{L}[ty''] = -\frac{d}{ds}\mathcal{L}[y''] = -\frac{d}{ds}\left(s^2 Y(s) - s y(0) - y'(0) \right)$

$= -2sY(s) - s^2 Y'(s) + y(0)$

$\mathcal{L}[ty'' + y' + ty] = 0$

$\Rightarrow -2sY(s) - s^2 Y'(s) + y(0) + sY(s) - y(0) - Y'(s) = 0$.

i.e. $(1+s^2)Y'(s) + sY(s) = 0$.

b> $\frac{Y'(s)}{Y(s)} = -\frac{s}{1+s^2} \Rightarrow \frac{dY}{Y} = -\frac{1}{2}\frac{ds^2}{1+s^2}$

$\Rightarrow Y(s) = c(1+s^2)^{-\frac{1}{2}}$ where $c =$ const.

c> $(1+s^2)^{-\frac{1}{2}} = s^{-1}(1+s^{-2})^{-\frac{1}{2}}$

$$= s^{-1} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} s^{-2k}$$

$$= s^{-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-k+1)}{k!} s^{-2k}$$

$$= s^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k (2k-1)!!}{2^k\, k!} s^{-2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2k-1)!!}{(2k)!!} s^{-2k-1}.$$

since $\mathcal{L}[t^k] = \frac{n!}{s^{n+1}}$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s^{2k+1}}\right] = \frac{t^{2k}}{(2k)!}$$

i.e. $\mathcal{L}^{-1}\left[(1+s^2)^{-\frac{1}{2}}\right] = \sum_{k=0}^{\infty} \frac{(-1)^k (2k-1)!!}{(2k)!!\,(2k)!} t^{2k}$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{((2k)!!)^2} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{2^{2k}(k!)^2}.$$

i.e. $y = c \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{2^{2k}(k!)^2} = c J_0(t).$

37. Sol:

By using $\mathcal{L}[t^n y] = (-1)^n \frac{d^n}{ds^n} Y(s)$

$$\mathcal{L}[y^{(n)}] = s^n Y(s) - s^{n-1} y(0) - \cdots - s\, y^{(n-2)}(0) - y^{(n-1)}(0).$$

we have: a> $Y' + s^2 Y = s$

b> $s^2 Y'' + 2s Y' - [s^2 + \alpha(\alpha+1)] Y = -1.$

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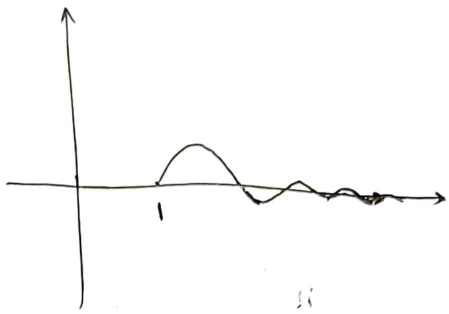
14. Sol:

a) $\mathcal{L}[y'' + \gamma y' + y] = \mathcal{L}[\delta(t-1)]$

$$\Rightarrow \quad s^2 Y(s) + \gamma s Y(s) + Y(s) = e^{-s}$$

when $\gamma = \frac{1}{2}$, it becomes:

$$(s^2 + \tfrac{1}{2}s + 1)\, Y(s) = e^{-s}$$



$$\Rightarrow \quad Y(s) = \frac{e^{-s}}{(s+\frac{1}{4})^2 + \frac{15}{16}}$$

$$\Rightarrow \quad y(t) = u_1(t)\left[e^{-\frac{1}{4}(t-1)} \cdot \frac{4}{\sqrt{15}} \cdot \sin\left(\frac{\sqrt{15}}{4}(t-1)\right)\right]$$

$$= \frac{4}{\sqrt{15}}\, u_1(t)\, e^{-\frac{t-1}{4}} \sin\left(\frac{\sqrt{15}}{4}(t-1)\right).$$

b) $$y'(t) = \frac{4}{\sqrt{15}}\, u_1(t) \left\{ -\frac{1}{4} e^{-\frac{t-1}{4}} \sin\left(\frac{\sqrt{15}}{4}(t-1)\right) + \frac{\sqrt{15}}{4} e^{-\frac{t-1}{4}} \cos\left(\frac{\sqrt{15}}{4}(t-1)\right)\right\}$$

$$= \frac{4}{\sqrt{15}}\, u_1(t)\, e^{-\frac{t-1}{4}} \left\{ -\frac{1}{4} \sin\left(\frac{\sqrt{15}}{4}(t-1)\right) + \frac{\sqrt{15}}{4} \cos\left(\frac{\sqrt{15}}{4}(t-1)\right)\right\}.$$

set $\theta := \arctan(-\sqrt{15})$, then

$$y'(t) = \frac{4}{\sqrt{15}}\, u_1(t)\, e^{-\frac{t-1}{4}} \sin\left(\frac{\sqrt{15}}{4}(t-1) + \theta\right)$$

$$y'(t) = 0 \Rightarrow \frac{\sqrt{15}}{4}(t-1) + \arctan(-\sqrt{15}) = \frac{\pi}{2}$$

i.e. $$t_1 = 1 + \frac{2\pi}{\sqrt{15}} + \frac{4\arctan(\sqrt{15})}{\sqrt{15}} \cong 2.3613,$$

$$y_1 = \frac{4}{\sqrt{15}} e^{-\frac{t_1-1}{4}} \cong 0.71153$$

c) $y = \frac{8\sqrt{7}}{21} u_1(t)\, e^{-\frac{t-1}{8}} \sin\left[\frac{3\sqrt{7}}{8}(t-1)\right]$; $t_1 \cong 2.4569$, $y_1 \cong 0.83351$.

d> Obs: $\mathcal{L}[y''+\gamma y'+y] = \mathcal{L}[\delta(t-1)]$

$$\Rightarrow (s^2+\gamma s+1)\,Y(s) = e^{-s}$$

$$\Rightarrow Y(s) = \frac{e^{-s}}{(s+\frac{\gamma}{2})^2+1-\frac{\gamma^2}{4}}$$

$$\Rightarrow y(t) = u_1(t)\,\frac{1}{\sqrt{1-\frac{\gamma^2}{4}}}\,e^{-\frac{\gamma}{2}(t-1)}\sin\left\{\sqrt{1-\frac{\gamma^2}{4}}\,(t-1)\right\}$$

$$y'(t)=0 \Rightarrow -\frac{\gamma}{2}\sin\left\{\sqrt{1-\frac{\gamma^2}{4}}\,(t-1)\right\} + \sqrt{1-\frac{\gamma^2}{4}}\cos\left(\sqrt{1-\frac{\gamma^2}{4}}\,(t-1)\right)=0.$$

take $\theta := \arctan\left[-\dfrac{2\sqrt{1-\frac{\gamma^2}{4}}}{\gamma}\right]$

$$\Rightarrow \sqrt{1-\frac{\gamma^2}{4}}\,(t-1) - \arctan\left(\frac{\sqrt{4-\gamma^2}}{\gamma}\right) = \frac{\pi}{2}$$

$$\Rightarrow \begin{cases} t_1 \nearrow & \text{as } \gamma \searrow. \\ y_1 \nearrow & \text{as } \gamma \searrow. \end{cases}$$

when $\gamma=0$, $t_1 = 1+\frac{\pi}{2}$, $y_1=1$.

25. Sol: omitted.

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8. Sol: $f(t) = \frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau\, d\tau$

9. Sol: $f(t) = \int_0^t e^{-(t-\tau)} \cos 2\tau\, d\tau$

10. Sol: $f(t) = \frac{1}{2}\int_0^t (t-\tau)\, e^{-(t-\tau)} \sin 2\tau\, d\tau$

16. Sol: $y = e^{-\frac{t}{2}}\cos t - \frac{1}{2}e^{-\frac{t}{2}}\sin t + \int_0^t e^{-\frac{(t-\tau)}{2}}\sin(t-\tau)\,[1-u_\pi(\tau)]\,d\tau$

20. Sol: $y = \frac{4}{3}\cos t - \frac{1}{3}\cos 2t + \frac{1}{6}\int_0^t [2\sin(t-\tau) - \sin 2(t-\tau)]\, g(\tau)\, d\tau.$

25.

$y'' + 2y' + 2y = f(t)$ $\quad y(0) = Y_0$

$y'(0) = Y_1$

$\lambda^2 + 2\lambda + 2 = 0$.

$\lambda_\pm = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$.

$\downarrow$

$\underline{y_1 = e^{-t}\cos t}$.

$\underline{y_2 = e^{-t}\sin t}$.

$y' =$

$y_p =$

$y = v_1(t)\, y_1 + v_2(t)\, y_2$.

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = e^{-t}\cos t(-e^{-t}\sin t + e^{-t}\cos t) - e^{-t}\sin t(-e^{-t}\cos t - e^{-t}\sin t)$$
$$= -e^{-2t}\sin t\cos t + e^{-2t}\cos^2 t + e^{-2t}\sin t\cos t + e^{-2t}\sin^2 t = e^{-2t}$$

$\Rightarrow$ $y_1 v_1' + y_2 v_2' = 0$.

$y_1' v_1' + y_2' v_2' = f(t)$.

$$\Rightarrow v_1' = \frac{-f(t)\, y_2(t)}{y_1 y_2' - y_1' y_2} = \frac{-f(t)\, e^{-t}\sin t}{e^{-2t}} = -f(t)\, e^{t}\sin t$$

$$v_2' = \frac{f(t)\, y_1(t)}{y_1 y_2' - y_1' y_2} = \frac{f(t)\, e^{-t}\cos t}{e^{-2t}} = f(t)\, e^{t}\cos t.$$

$$\Rightarrow v_1 = \int -f(t)\, e^{t}\sin t\, dt, \quad v_2 = \int f(t)\, e^{t}\cos t\, dt.$$

$$y_p(t) = -\int f(t)\, e^{t}\sin t\, dt \cdot e^{-t}\cos t + \int f(t)\, e^{t}\cos t\, dt\, e^{-t}\sin t.$$

~~$\int_0^t f(\tau) e^{\tau} \ldots d\tau$~~ $\int e^{-(t-\tau)} f(\tau)\sin(t-\tau)\,d\tau \Big|_{\tau=t} - \int \ldots d\tau \Big|_{\tau=0}$

$= \int f(t)\; 0$

$s^2 Y - 2s\, y(0) - y'(0) + 2sY + 2Y = F(s)$

$Y(s) = \frac{F(s)}{s^2 + 2 + 2s}$, $(s+1)^2 + 1$ $\Rightarrow y(t) = f * (\sin t \cdot e^{-t}) = \int_0^t f(\tau)\, e^{-(t-\tau)}\sin(t-\tau)\,d\tau$.

$$= -\left(\int_0^t f(\tau) e^{\tau}\sin\tau\, d\tau + C_1\right) e^{-t}\cos t + \left(\int_0^t f(\tau)\, e^{\tau}\cos\tau\, d\tau + C_2\right) e^{-t}\sin t.$$

$y(0) = -C_1 = 0$.

$y' = -(f(t) e^{t}\sin t)\, e^{-t}\cos t$ ~~$-(e^{-t}\cos t \mp e^{-t}\sin t)$~~ $(\int \ldots + C_1)$

~~$f(0)$~~ $(-e^{-t}\sin t + e^{-t}\cos t)(\int_0^t \ldots + C_2) = C_2 = 0$

$$y(t) = -\int_0^t f(\tau)\, e^{\tau} \sin\tau \, d\tau \cdot e^{-t}\cos t + \left(\int_0^t f(\tau)\, e^{\tau}\cos\tau \, d\tau \; e^{-t}\sin t\right)$$

$$= -\int_0^t f(\tau)\, e^{\tau-t} \sin\tau \cos t \, d\tau + \int_0^t f(\tau)\, e^{\tau-t} \cos\tau \sin t \, d\tau.$$

$$= \int_0^t f(\tau)\, e^{\tau-t} (\cos\tau \sin t - \sin\tau \cos t)\, d\tau$$

$$= \int_0^t f(\tau)\, e^{\tau-t} \sin(t-\tau)\, d\tau$$

$$= \int_0^t f(\tau)\, e^{-(t-\tau)} \sin(t-\tau)\, d\tau.$$

(b). $f(t) = \delta(t-\pi)$

$\left(\int_0^t \delta(u-\pi) f(u)\, du = f(\pi)\cdot u(t-\pi)\right)$

$$y(t) = \int_0^t \delta(\tau-\pi)\, e^{-(t-\tau)} \sin(t-\tau)\, d\tau$$

$$= u(t-\pi)\cdot e^{-(t-\pi)} \sin(t-\pi)$$

(c). $$Y(s) = \frac{e^{-\pi s}}{s^2+2+2s} = \frac{e^{-\pi s}}{(s+1)^2+1} = e^{-\pi s} \mathcal{L}(\underbrace{\sin t \cdot e^{-t}}_{f})$$

$$y(t) = \cancel{(s+\ldots)}\; f(t-\pi)\, u(t-\pi) = u(t-\pi)\, e^{-(t-\pi)} \sin(t-\pi).$$